International Journal of Engineering, Science and Mathematics
Vol. 8 Issue 6, June 2019,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Bernstein L^P Type Inequality of Some Class of Polynomials

Jahangeer Habibullah Ganai [1] and Anjna Singh[2]
[1,2]Department of Mathematical Sciences A.P.S. University, Rewa (M.P.)
486003 India,
Govt. Girls P.G. College Rewa,(M.P)

Abstract. In the present paper we will discuss Bernstein's classical theorem for a polynomial F of degree m, $\max_{|z|=1}|F'(t)| \le m \max_{|z|=1}|F(t)|$. We will give some related results for a polynomial F holds the conditions

$F'(0) = F''(0) = \dots = F^{(m-1)}(0) = 0$ and $F(t) \ne 0$ for $|t| < l$, where $l \ge 1$. We will give L^P

inequalities valid for $0 \le r \le \infty$.

Keywords. Minkowski's Inequality, Erdos Conjectured, Malik Generalized Theorem

Introduction

Let $\mathcal{F}_m$ be the linear space of polynomials over the complex field of degree less than or equal to m. For F$\in \mathcal{F}_m$, define

$$\|F\|_0 = \exp\left(\frac{1}{2}\int_0^{2\pi} \log\left|F(e^{i\psi})\right| d\psi\right) \tag{1}$$

$$\|F\|_f = \left(\frac{1}{2}\int_0^{2\pi} \left|F(e^{i\psi})\right|^r d\psi\right)^{1/r} \quad for \quad 0 < r < \infty$$

(2)

$$\|F\|_\infty = \max_{|t|=1}|F(t)|.$$

(3)

$$Notice \quad that \quad \|F\|_0 = \lim_{f\to 0^+}\|F\|_f \quad and \quad \|F\|_\infty = \lim_{f\to\infty}\|F\|_f. \quad For \quad 1 \le r \le \infty.$$

$\|\cdot\|$ is a norm and therefore $\mathcal{F}_m$ is a normed linear space under $\|\cdot\|_f$. However, for $0 \le r \le 1$, $\|\cdot\|_f$ does not satisfy the triangle inequality and is therefore not a norm this follows from Minkowski's inequality see [3].

Bernstein's well knowing result relating the supremum norm of a polynomial and its derivative states that if F$\in \mathcal{F}_m$ then $\|F'\|_\infty$ then $\|F'\|_\infty \le m\|F\|_\infty$ [9]. This inequality reduces to equality if and only if $F(t) = \beta t^m$ for some complex constant β .Erdos conjectured and Lax proved [7].

Theorem 1. If F$\in \mathcal{F}_m$ and $F(t) \ne 0 \quad for |t| < 1, \quad then$

$$\|F'\|_\infty \le \frac{m}{2}\|F\|_\infty$$

(4)

Malik generalized Theorem 1 and proved [4]

Theorem 2. If $F \in \mathcal{F}_m$ and $F(t) \neq 0 \quad for|t| < l \quad where \quad l \geq 1, then$

$$\|F'\|_\infty \leq \frac{m}{1+l}\|F\|_\infty$$

(5)

Of course. Theorem 1 follows from Theroem2 when l=1. Chan and Malik [3] introduced the class of polynomials of the form $F(t) = b_0 + \sum_{v=m}^{n} b_v t^v$. We denote the linear space of all such polynomials as $\mathcal{F}_{n,m}$. we Notice that $\mathcal{F}_{n,1} = \mathcal{F}_n$ Chan and Malik presented the following result [3].

Theorem 3. If $F \in \mathcal{F}_m$ and $F(t) \neq 0 \quad for|t| < l$ where $l \geq 1,$ then

$$\|F'\|_\infty \leq \frac{m}{1+l^m}\|F\|_\infty \tag{6}$$

Qazi, independently of Chan and Malik, presented the following result which includes Theorem3 [8]

Theorem 4. If $F(t) = b_0 + \sum_{v=m}^{n} b_v t^v \in F_{n,m} \quad and \quad F(t) \neq 0$ for $|t| < l \quad where l \geq 1,$ then

$$\|F'\|_\infty \leq \frac{m}{1+J_0}\|F\|_\infty \tag{7}$$

Where

$$J_0 = l^{m+1}\left|\frac{m|b_m|l^{m-1} + m|b_0|}{m|b_0| + m|b_m|l^{m+1}}\right| \tag{8}$$

Since $m|b_m|l^m \leq m|b_0|$. Theorem 4 Implies Theorem 3.

Zygmund [11] extended Bernstein's result to L^P norms. DeBruijin [6] extended theorem 1 to L^P norms by showing.

Theorem 5. If $F \in \mathcal{F}_m$ and $F(t) \neq 0$ for $|t| < 1,$ then for $1 \leq r \leq \infty$

$$\|F'\|_f \leq \frac{m}{\|1+t\|_f}\|F\|_f \tag{9}$$

Of course theoem 5 reduces to theorem 1 with $r = \infty$. Rahman and schmeisser [8] proved that theorem5 in fact holds for $0 \leq r \leq \infty$. The purpose of this paper is to show that theorem3 and theorem4 can be extended to L^P inequalities where $0 \leq r \leq \infty$.

STATEMENT

Theorem 6. If $F(t) = b_0 + \sum_{v=m}^{n} b_v t^v \in F_{n,m} \quad and \quad F(t) \neq 0$ for $|t| < l$ where $l \geq 1,$ then for $0 \leq r \leq \infty$

$$\|F'\|_f \leq \frac{m}{\|J_0 + t\|_f}\|F\|_f$$

(10)

Where J_0 is as given in Theorem4. With $r = \infty$, theorem 2 reduces to theorem 4. As mentioned.

Corollary 1. If $F \in \mathcal{F}_m$ and $F(t) \neq 0$ for $|t| < l$ where $l \geq 1,$ then for $0 \leq r \leq \infty$

$\|F'\|_f \leq \frac{m}{\|l^m + t\|_f}\|F\|_f$ with $r = \infty$, Corollary 1 reduces to theorem3 of special interest is the fact theorem 2 and corollary 1 holds for L^P norms for all $1 \leq r \leq \infty$. In Particular, we have

Corollary 2. If $F \in \mathcal{F}_{n,m}$ and $F(t) \neq 0$ for $|t| < l$ where $l \geq 1$, then for $1 \leq r \leq \infty$

$$\|F'\|_f \leq \frac{m}{\|l^m + t\|_f}\|F\|_f \tag{11}$$

With m=1, Corollary 2 yields an L^P version of theorem 2 with $r = \infty$, Corollary 2 reduces to theorem 3 with m=1 and $r = \infty$ Corollary 2 reduces to theorem 2. Finally with m=1, $r = \infty$ and l=1, Corollary 2 reduces to Theorem1.

LEEMAS

We need the following leemas for the proof of our theorem.

Leema1. If the polynomial F(t) of degree m has no roots in the circular domain C and if $\sigma \in D$ then $(\sigma - t)F(t) + mF(t) \neq 0 \quad for \quad t \in D$

Leema 1 is due to Laguerre [5].

Definition for $\beta = (\beta_0, \ldots, \beta_m) \subset D^{m+1} \quad and \quad F(t) = \sum_{v=0}^{m} D_v t^v$

$$\wedge_\beta F(t) = \sum_{v=0}^{m} \beta_v D_v t^v$$

The Operator $\wedge_\beta$ is said to be admissiable it if preserves one of the following properties.

(1) F(t) has all it zeros in $\{t \in D : |t| \leq 1\}$

(2) F(t) has all it zeros in $\{t \in D : |t| \geq 1\}$

The proof of leema 3 was given by Arestov [1]

Leema 2. Let $\phi(t) = \psi(\log t)$ where ψ is a convex non decreasing function on R Then for all $F(t) \in F_n$ and each admissible operator $\wedge_\beta$

$$\int_0^{2\pi} \phi(|\wedge_\beta F(e^{i\theta})|) d\theta \leq \int_0^{2\pi} \phi(D(\beta, m)|F(e^{i\theta})|) d\theta$$

(12)

Where $D(\beta, m) = \max(|\beta_0|, |\beta_m|)$ Qazi proved [5]

Leema3. If $F(t) = D_0 + \sum_{v=m}^{n} D_v t^v$ has no zeros in $|t| < l, l \geq 1 \quad then \quad for \quad |t| = 1$

$$L_m|F'(t)| \leq J_0|F'(t)| \leq |R'(t)|$$

Where $R(t) = t^n \overline{F\left(\frac{1}{z}\right)}$ and J_0 is as defined in theorem4.

By lemma3 we have $mF(t) - (t - \sigma)F'(t) \neq 0$ for $|t| \leq 1, \sigma \leq 1$ Therefore setting $\sigma = -te^{-i\beta}, \quad \beta \in R$ the operator $\wedge$ defined by

$$\wedge F(t) = (e^{i\beta} + 1)tF'(t) - me^{i\beta}F(t)$$

Is admissible and so by Lemma 3 with $\psi(x) = e^{fx}$

$$\int_0^{2\pi} \left|(e^{i\beta}+1)\frac{dF(e^{i\theta})}{d\theta} - ime^{i\beta}F(e^{i\beta})\right|^f d\theta \le m^f \int_0^{2\pi} |F(e^{i\theta})|^f d\theta$$

For $r > 0$

$$\int_0^{2\pi} \left|(\frac{dF(e^{i\theta})}{d\theta} + e^{i\beta}\left[\frac{dF(e^{i\theta})}{d\theta} - imF(e^{i\theta})\right]\right|^f d\theta \le m^f \int_0^{2\pi} |F(e^{i\theta})|^f d\theta$$

This gives

$$\int_0^{2\pi}\int_0^{2\pi} \left|(\frac{dF(e^{i\theta})}{d\theta} + e^{i\beta}\left[\frac{dF(e^{i\theta})}{d\theta} - imF(e^{i\theta})\right]\right|^f d\theta d\beta \le 2\pi m^F \int_0^{2\pi} |F(e^{i\theta})|^f d\theta$$

(13)

$$\int_0^{2\pi}\int_0^{2\pi} \left|(\frac{dF(e^{i\theta})}{d\theta} + e^{i\beta}\left[\frac{dF(e^{i\theta})}{d\theta} - imF(e^{i\theta})\right]\right|^f d\theta d\beta$$

$$= \int_0^{2\pi} \left|\frac{dF(e^{i\theta})}{d\theta}\right|^f \int_0^{2\pi} \left|1 + e^{i\beta}\left(\frac{dF(e^{i\theta})/d\theta - imF(e^{i\theta})}{dF(e^{i\theta})/d\theta}\right)\right|^f d\beta d\theta$$

$$= \int_0^{2\pi} \left|\frac{dF(e^{i\theta})}{d\theta}\right|^f \int_0^{2\pi} \left|e^{i\beta} + \left(\frac{dF(e^{i\theta})/d\theta - imF(e^{i\theta})}{dF(e^{i\theta})/d\theta}\right)\right|^f d\beta d\theta$$

$$= \int_0^{2\pi} \left|\frac{dF(e^{i\theta})}{d\theta}\right|^f \int_0^{2\pi} \left|e^{i\beta} + \left|\frac{Q'(e^{i\theta})}{F'(e^{i\theta})}\right|\right|^f d\beta d\theta$$

$$\ge \int_0^{2\pi} \left|\frac{dF(e^{i\theta})}{d\theta}\right|^f \int_0^{2\pi} \left|e^{i\beta} + J_0\right|^f d\beta d\theta$$

(14)

By the fact that $|e^{i\beta} + r|$ is an increasing function of r for $r \ge 1.$ Thus combining (13) and(14) we get

$$\left(\int_0^{2\pi} \left|(\frac{dF(e^{i\theta})}{d\theta}\right|^f d\theta\right)\left(\int_0^{2\pi} |e^{i\beta} + J_0| d\beta\right) \le 2\pi m^F \int_0^{2\pi} |F(e^{i\theta})|^f d\theta$$

(15)

From which the theorem follows for $0 \le r \le \infty$. This results holds good for r=0 and $r = \infty$ by letting $r \to 0^+ \quad and \quad r \to \infty$.

REFERENCES

1. A. Zygmund, ‘‘Trigonometric Series,’’ 2nd rev. ed., Vols. I, II, Cambridge Univ. Press, Cambridge, UK, 1959.
2. E. Laguerre, ‘‘Oeuvres,’’ Vol. 1; *Nou¤elles Ann. Math.* **17**, No. 2 1878..
3. H. Royden, ‘‘Real Analysis,’’ 3rd ed., Macmillan Co., New York, 1988.
4. M. Malik, On the derivative of a polynomial, *J. London Math. Soc.* **1**, No. 2 1969., 57]60.
5. M. Qazi, On the maximum modulus of polynomials, *Proc. Amer. Math. Soc.* **115** 1992., 337]343.
6. N. DeBruijn, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensch. Proc.* **50**, 1947., 1265]1272.
7. P. Lax, Proof of a conjecture of P.Erdos on the derivative of a polynomial, *Bull,Amer.math.Soc*.**50** (1994),509-513
8. Q. Rahman and G. Schmeisser, L^p inequalities for polynomials, *J. Approx. Theory* **53** 1988., 26]32.
9. S. Bernstein, ‘‘Lec¸ons sur les proprietes´´ extremales´ et la meilleure approximation des
10. T. Chan and M. Malik, On Erdos¨]Lax theorem, *Proc. Indian Acad. Sci.* **92** 1983., 191]193.
11. V. Arestov, On integral inequalities for trigonometric polynomials and their derivatives, *Math. USSR-Iz¤.* **18** *(*1982), 1-17.